11. Probability

- **Certain events:** Events which are definite to happen.

  For example, the day after Saturday will be Sunday or the sun will rise from the east.

- **Impossible events:** Events which are impossible to happen.

  For example, March comes before February in a year, the apple goes up when dropped from the tree.

- **Matter of Chance:** Results of events which can not be known before they happen.

  In a cricket match, India will win or it will rain tomorrow.

- **Probability** is the measure or estimation of likelihood of happening of an event in a particular way.

- Some of the terms related to probability are:
  - **Experiment**: When an operation is planned and done under controlled conditions, it is known as an experiment. For example, tossing a coin, throwing a die etc., are all experiments.
  - **Outcomes**: Different results obtained in an experiment are known as outcomes. For example, on tossing a coin, if the result is a head, then the outcome is a head; if the result is a tail, then the outcome is a tail.
  - **Random**: An experiment is random if it is done without any conscious decision. For example, drawing a card from a well-shuffled pack of playing cards is a random experiment if it is done without seeing the card.
  - **Trial**: A trial is an action or an experiment that results in one or several outcomes. For example, if a coin is tossed five times, then each toss of the coin is called a trial.
  - **Sample space**: The set of all possible outcomes of an experiment is called the sample space. It is denoted by the letter 'S'. Sample space in the experiment of tossing a coin is {H, T}.
  - **Event**: The event of an experiment is one or more outcomes of the experiment. For example, tossing a coin and getting a head or a tail is an event.

- The outcomes of an experiment having the same chances of occurrence are known as equally-likely outcomes. For example, if we toss a coin, then the possible outcomes are head or tail, and both of them have an equal chance of occurring. So, these are equally-likely outcomes.

- When the outcomes of the experiment are equally-likely, the probability of an event is given by:

  $$\frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}}$$

- **The probability of occurrence of any event always lies between 0 and 1.**

  For example, a bag contains one green, one red, one blue, and one black ball. When a ball is drawn, it can be any of the four balls.

  The probability of drawing a red ball = $\frac{1}{4}$

Here, $\frac{1}{4}$ is greater than 0 but less than 1.

- **The probability of such an event which has no possibility to occur is 0**.

  For example, there is no possibility of drawing a green pen from the box containing blue and black pens only. In this case, the probability of drawing a green pen is 0.

- **The probability of such an event which is sure to occur is 1.**

  For example, if there is a box containing only blue pens, then the probability of drawing a blue pen is 1 because the pen drawn will always be blue.

- **Algebra of events**
  - **Complementary event:** For every event A, there corresponds another event A' called the complementary event to A. It is also called the event 'not A'.

  $A' = \{\omega : \omega \in S \text{ and } \omega \notin A\} = S - A.$

  - **The event '*A* or *B*':** When sets A and B are two events associated with a sample space, then the set $A \cup B$ is the event 'either A or B or both'.

  That is, event 'A or B' $= A \cup B = \{\omega: \omega \in A \text{ or } \omega \in B\}$

  - **The event '*A* and *B*':** When sets A and B are two events associated with a sample space, then the set $A \cap B$ is the event 'A and B'.

  That is, event 'A and B' $= A \cap B = \{\omega: \omega \in A \text{ and } \omega \in B\}$

  - **The event '*A* but not *B*':** When sets A and B are two events associated with a sample space, then the set $A - B$ is the event 'A but not B'.

  That is, event 'A but not B' $= A - B = A \cap B' = \{\omega: \omega \in A \text{ and } \omega \notin B\}$

Example: Consider the experiment of tossing 2 coins. Let A be the event 'getting at least one head' and B be the event 'getting exactly two heads'. Find the sets representing the events

(i) complement of 'A or B'

(ii) A and B

(iii) A but not B

Solution:
Here, S = {HH, HT, TH, TT}
A = {HH, HT, TH}, B = {HH}

(i) A or $B = A \cup B$ = {HH, HT, TH}

Hence, complement of A or B = (A or B)' = $(A \cup B)' = U - (A \cup B)$ = {TT}

(ii) A and $B = A \cap B$ = {HH}

(iii) A but not $B = A - B$ = {HT, TH}

- **Mutually Exclusive Events**

Two events, A and B, are called mutually exclusive events if the occurrence of any one of them excludes the occurrence of the other event i.e., if they cannot occur simultaneously.

In this case, sets A and B are disjoint i.e., $A \cap B = \emptyset$

If $E_1, E_2, \dots E_n$ are n events of a sample space S, and if

$\bigcup_{i=1}^{n} E_i = S$, $E_1 \cup E_2 \cup \dots \cup E_n = \cup_{i=1}^{n} E_i = S$ then
$E_1, E_2, \dots E_n$ are called mutually exclusive and exhaustive events.
In other words, at least one of $E_1, E_2, \dots E_n$ necessarily occurs whenever the experiment is performed.

The events $E_1, E_2, \dots E_n$, i.e., n events of a sample space (S) are called mutually exclusive and exhaustive events if

$E_i \cap E_i = \phi$ for $i \neq j$ i.e., events E_i and E_j are pairwise disjoint, and

$\cup_{i=1}^{n} = S$

Example: Consider the experiment of tossing a coin twice. Let A and B be the event of "getting at least one head" and "getting exactly two tails" respectively. Are the events A and B mutually exclusive and exhaustive?
Solution:
Here, S = {HH, HT, TH, TT}
A = {HH, HT, TH}
B = {TT}
Now, $A \cap B = \emptyset$ and $A \cup B$ = {HH, HT, TH, TT} = S
Thus, A and B are mutually exclusive and exhaustive events.

- The number P (ω_i) i.e., the probability of the outcome ω_i, is such that
  - $0 \leq P(\omega_i) \leq 1$
  - $\sum P(\omega_i) = 1$ for all $\omega_i \in S$
  - For any event A, $P(A) = \sum P(\omega_i)$ *for all* $\omega_i \in A$
- For a finite sample space S, with equally likely outcomes, the probability of an event A is denoted as P (A) and it is given by

$$P(A) = \frac{n(A)}{n(S)},$$

- Where, $n(A)$ = Number of elements in set A and $n(S)$ = Number of elements in set S
  - If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

  - If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

  - If A is any event, then

$$P(A') = 1 - P(A)$$

Example: Consider the experiment of tossing a die. Let A be the event "getting an even number greater than 2" and B be the event "getting the number 4". Find the probability of

(i) getting an even number greater than 2 or the number 4

(ii) getting a number, which is not the number 4, on the top face of the die

Solution: Here, $S = \{1, 2, 3, 4, 5, 6\}$
$A = \{4, 6\}$, $B = \{4\}$
$A \cap B = \{4\}$
$p(A) = \frac{2}{6}$, $p(B) = \frac{1}{6}$, $p(A \cap B) = \frac{1}{6}$

(i) Required probability = $P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B)$$

$$= \frac{2}{6} + \frac{1}{6} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

(ii) $P(B) = \frac{1}{6}$

$\therefore$ P (not B) = 1 – P (B) $= 1 - \frac{1}{6} = \frac{5}{6}$

Hence, the required probability of not getting number 4 on the top face of the die is $\frac{5}{6}$.

Example: 20 cards are selected at random from a deck of 52 cards. Find the probability of getting at least 12 diamonds.

Solution: 20 cards can be selected at random from a deck of 52 cards in ${}^{52}C_{20}$ ways. Hence, Total possible outcomes = ${}^{52}C_{20}$

P (at least 12 diamonds) = P (12 diamonds or 13 diamonds)
= P (12 diamonds) + P (13 diamonds)

$$= \frac{{}^{13}C_{12} \times {}^{39}C_8}{{}^{52}C_{20}} + \frac{{}^{13}C_{13} \times {}^{39}C_7}{{}^{52}C_{20}}$$

$$= \frac{13 \times {}^{39}C_8}{{}^{52}C_{20}} + \frac{{}^{39}C_7}{{}^{52}C_{20}}$$

$$= \frac{13 \times {}^{39}C_8 + {}^{39}C_7}{{}^{52}C_{20}}$$

$$= \frac{13 \times \frac{39!}{31! \times 8!} + \frac{39!}{32! \times 7!}}{{}^{52}C_{20}}$$

$$= \frac{13 \times \frac{39!}{31! \times 8!} + \frac{39! \times 8}{32 \times 31! \times 7! \times 8}}{{}^{52}C_{20}}$$

$$= \frac{13 \times \frac{39!}{31! \times 8!} + \frac{8}{32} \times \frac{39!}{31! \times 8!}}{{}^{52}C_{20}}$$

$$= \frac{\frac{53}{4} \times \frac{39!}{31! \times 8!}}{{}^{52}C_{20}}$$

$$= \frac{53}{4} \times \frac{{}^{39}C_8}{{}^{52}C_{20}}$$

- **Complementary events**

For an event E such that $0 \le P(E) \le 1$ of an experiment, the event $\bar{E}$ represents 'not E' , which is called the complement of the event E. We say, E and $\bar{E}$ are **complementary** events.

$P(E) + P(\bar{E}) = 1$

$\Rightarrow P(\bar{E}) = 1 - P(E)$

Example:
A pair of dice is thrown once. Find the probability of getting a different number on each die.
Solution:
When a pair of dice is thrown, the possible outcomes of the experiment can be listed as:

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The number of all possible outcomes = 6 × 6 = 36
Let E be the event of getting the same number on each die.
Then, $\bar{E}$ is the event of getting different numbers on each die.

Now, the number of outcomes favourable to E is 6.
$\therefore P(\bar{E}) = 1 - P(E) = 1 - \frac{6}{36} = \frac{5}{6}$

Thus, the required probability is $\frac{5}{6}$.

1. For any two events *A* and *B* of a sample space *S*,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

2. For two events *A* and *B*, there may be two possibilities as follows:

(i) If *A* and *B* are mutually exclusive events then

$P(A \cup B) = P(A) + P(B)$

(ii) If *A* and *B* are mutually exclusive and exhaustive events then

$P(A) + P(B) = 1$

- If *E* and *F* are two events associated with the sample space of a random experiment, then the conditional probability of event *E*, given that *F* has already occurred, is denoted by P(*E*/*F*) and is given by the formula:

$P(E/F) = \frac{P(E \cap F)}{P(F)}$, where $P(F) \neq 0$

Example:

A die is rolled twice and the sum of the numbers appearing is observed to be 7. What is the conditional probability that the number 3 has appeared at-least once?

Solution:

Let E: Event of getting the sum as 7 and F: Event of appearing 3 at-least once

Then ,$E = \{(1, 6), (2, 5)(3, 4), (4, 3), (5, 2), (6, 1)\}$ and

$$F = \{(1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (6, 3), (3, 1), (3, 2), (3, 4), (3, 5), (3, 6)\}$$

$\therefore E \cap F = \{(3, 4), (4, 3)\}$

$n(E) = 6, n(F) = 11$ and $n(E \cap F) = 2$

$$\mathrm{P}\left(\frac{F}{E}\right) = \frac{\mathrm{P}(E \cap F)}{\mathrm{P}(E)} = \frac{n(E \cap F)}{n(E)} = \frac{2}{6} = \frac{1}{3}$$

- If E and F are two events of a sample space S of an experiment, then the following are the properties of conditional probability:
  - $0 \leq \mathrm{P}(E/F) \leq 1$
  - $\mathrm{P}(F/F) = 1$
  - $\mathrm{P}(S/F) = 1$
  - $\mathrm{P}(E'/F) = 1 - \mathrm{P}(E/F)$
  - If A and B are two events of a sample space S and F is an event of S such that $\mathrm{P}(F) \neq 0$, then
    - $\mathrm{P}((A \cup B)/F) = \mathrm{P}(A/F) + \mathrm{P}(B/F) - \mathrm{P}((A \cap B)/F)$
    - $\mathrm{P}((A \cup B)/F) = \mathrm{P}(A/F) + \mathrm{P}(B/F)$, if the events A and B are disjoint.

- **Multiplication theorem of probability:** If E, F, and G are events of a sample space S of an experiment, then
  - $\mathrm{P}(E \cap F) = \mathrm{P}(E). \mathrm{P}(F/E)$, if $\mathrm{P}(E) \neq 0$
  - $\mathrm{P}(E \cap F) = \mathrm{P}(F). \mathrm{P}(E/F)$, if $\mathrm{P}(F) \neq 0$
  - $\mathrm{P}(E \cap F \cap G) = \mathrm{P}(E). \mathrm{P}(F/E). \mathrm{P}(G/(E \cap F)) = \mathrm{P}(E). \mathrm{P}(F/E). \mathrm{P}(G/EF)$

- Two events E and F are said to be independent events, if the probability of occurrence of one of them is not affected by the occurrence of the other.
- If E and F are two independent events, then
  - $\mathrm{P}(F/E) = \mathrm{P}(F)$, provided $\mathrm{P}(E) \neq 0$
  - $\mathrm{P}(E/F) = \mathrm{P}(E)$, provided $\mathrm{P}(F) \neq 0$
- If three events A, B, and C are independent events, then
  $\mathrm{P}(A \cap B \cap C) = \mathrm{P}(A). \mathrm{P}(B). \mathrm{P}(C)$

- If the events E and F are independent events, then
  - E' and F are independent
  - E' and F' are independent
- A set of events $E_1, E_2, \ldots E_n$ is said to represent a partition of the sample space S, if
  - $E_i \cap E_j = \varnothing, i \neq j, i, j = 1, 2, 3, \ldots n$
  - $E_1 \cup E_2 \cup \ldots \cup E_n = S$
  - $P(E_i) > 0, \forall i = 1, 2, 3, \ldots n$

- **Bayes' Theorem:** If $E_1, E_2, \ldots E_n$ are n non-empty events, which constitute a partition of sample space S, then

$$P(E_i / A) = \frac{P(E_i)P(A/E_i)}{\sum_{j=1}^{n} P(E_j)P(A/E_j)}, i = 1, 2, 3, \ldots n$$

Example:

There are three urns. First urn contains 3 white and 2 red balls, second urn contains 2 white and 3 red balls, and third urn contains 4 white and 1 red balls. A white ball is drawn at random. Find the probability that the white ball is drawn from the third urn?

Solution:

Let E_1, E_2 and E_3 be the events of choosing the first second and third urn respectively.

Then, $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Let A be the event that a white ball is drawn.

Then, $P\left(\frac{A}{E_1}\right) = \frac{3}{5}, P\left(\frac{A}{E_2}\right) = \frac{2}{5} \text{ and } P\left(\frac{A}{E_3}\right) = \frac{4}{5}$

By the theorem of total probability,

$$P(A) = P(E_1) \times P\left(\frac{A}{E_1}\right) + P(E_2) \times P\left(\frac{A}{E_2}\right) + P(E_3) \times P\left(\frac{A}{E_3}\right)$$

$$= \frac{1}{3} \times \frac{3}{5} + \frac{1}{3} \times \frac{2}{5} + \frac{1}{3} \times \frac{4}{5}$$

$$= \frac{3}{5}$$

By Bayes' theorem,

probability of getting the ball from third urn given that it is white

$$= P\left(\frac{E_3}{A}\right) = \frac{P(E_3)P\left(\frac{A}{E_3}\right)}{P(A)} = \frac{\frac{1}{3} \times \frac{4}{5}}{\frac{3}{5}} = \frac{4}{9}$$